

Jeans Instability in the linearized Burnett regime

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Abstract

Jeans instability is derived for the case of a low density self-gravitating gas beyond the Navier-Stokes equations. The Jeans instability criterium is shown to depend on a Burnett coefficient if the formalism is taken up to fourth order in the wave number. It is also shown that previously known viscosity corrections to the Jeans wave-number are enhanced if the full fourth order formalism is applied to the stability analysis.

1 Introduction

The physics of low density systems is relevant in the description of thermodynamic processes which presumably occurred in the universe some time between its age of 10^6 sec, temperature $T = 1$ KeV and the stage after matter decoupled from radiation when the matter density reached values of $\sim 10^{-18} \frac{g}{cm^3}$ and $T = 1$ eV. It is also known that these types of regimes prevail in many other astrophysical systems [1]. Since a hydrodynamical, Navier-Stokes, description of a low density fluid breaks down for small Knudsen numbers, it is desirable to go beyond this regime and analyze the fluid with the tools that statistical physics provide as the particle collision frequency decreases. One simple tool useful for this purpose is, on the one hand, the phenomenological formalism now known as Linear Irreversible Thermodynamics (LIT), which provides a good grasp about how non-equilibrium processes evolve in time [2]. On the other hand, kinetic

theory provides a solid mesoscopic foundation for this theory, specially if we want to deal with hydrodynamics beyond the Navier-Stokes regime [3] [4].

The relevance of transport coefficients in cosmological and astrophysical problems has already been widely recognized [1] [5] [6]. Nevertheless, the Burnett equations [3] [4] have rarely been applied to structure formation problems in a phenomenological context, although the Burnett regime itself has been addressed in the context of transport theory in special relativity [7] [8] [9]. One possible motivation for the study of the Burnett regime in cosmological situations is the well-known relation between wave scattering processes, density fluctuations and dissipative effects [10] [11] [12]. Dynamic structure factors, possibly relevant for describing CMB distortions, can be derived from the hydrodynamical Burnett approach, as it has already been done for collisionless plasmas [13]. The use of these structure factors has appeared to be useful in the study of scattering laws to describe the Sunyaev-Zel'dovich effect and can, in principle, be applied to study density-density correlation functions to deal with hydrodynamical instabilities [14] or temperature-temperature correlation functions, a problem that remains to be studied via a thermodynamical approach. For the background microwave radiation this is done at present using the well-known multipole expansion. In this work we analyze the solutions to the phenomenological linearized Burnett equations in the context of Jeans instability theory, generalizing previous work [15] [16]. The main reason for doing this is that the introduction of the Burnett regime allows to take into account all k^4 order terms in the dispersion relation. Furthermore, the general tenets behind this formalism are worth emphasizing in view of its wide range of applicability [17]. To accomplish this task, the paper is divided as follows: section 2 reviews the basic phenomenological equations governing the Burnett regime. Section three is dedicated to the analysis of the dispersion relation obtained from the linearized formalism. Final remarks about the implications of the Burnett regime in structure formation are included in the last section.

2 Basic formalism

The starting point for our calculation concerns the method whereby for a simple fluid which is not in equilibrium we can apply the adequate information to transform the well-known conservation equations for mass, momentum and energy into a complete set. For this purpose, we assume that the fluid is isothermal and, therefore, the energy equation may be ignored. This leaves us with the continuity and momentum equations. The second one has the form,

$$\rho \frac{du^i}{dt} + \frac{\partial \tau^{ij}}{\partial x^j} = f^i \quad (1)$$

where f^i is the external force per unit of volume. To relate the stress tensor τ^{ij} with the mass density ρ and the local velocity u^i , one must appeal to experiment or to kinetic theory. The main problem here is that Eq. (1) together with the continuity equation involve 4 variables, ρ and u^i . Thus, there are four equations

but ten unknowns assuming that the fluid is isotropic and, therefore, τ^{ij} is symmetric. So, we require knowledge of the constitutive equations relating τ^{ij} to the set of local variables. When this procedure is accomplished, keeping all those terms which are at most quadratic in the gradients of ρ and u^i we refer to it as the Burnett regime. How to accomplish this using kinetic theory is now briefly described. We assume that the fluid is a dilute inert gas whose dynamics is governed by the Boltzmann equation [3] [4]. For such a model, it is well known that through the Chapman-Enskog method one can obtain the solutions to the equation which are consistent with the macroscopic hydrodynamic equations. This is achieved by expanding the single particle distribution function in power series of the Knudsen parameter ϵ , which is a measure of the spatial gradients present in the gas, responsible for its deviations from the equilibrium states. When this procedure is carried over, one obtains, to order zero in ϵ the Euler regime, and to first order in ϵ the Navier-Stokes regime with zero bulk viscosity, so that

$$\tau_{NS}^{ij} = p\delta^{ij} - 2\eta(\sigma^{ij} - \frac{1}{3}\theta\delta^{ij}) \quad (2)$$

where σ^{ij} is the symmetric traceless part of the velocity gradient, η is the shear viscosity, p is the local pressure, δ^{ij} is Kronecker's delta and $\theta \equiv \frac{\partial u^i}{\partial x^i}$. To next order in ϵ one gets the Burnett regime, in which τ^{ij} becomes a rather complicated expression [4] containing both, nonlinear as well as linear terms in the velocity gradient. If we further assume that the gas is slightly deviating from its local equilibrium state, the *linearized* Burnett contribution to the stress tensor τ_B^{ij} is then given by:

$$\tau_B^{ij} = \varpi_2 \frac{\eta^2}{p_o} \left[\frac{\partial(\frac{1}{\rho_o} f^i)}{\partial x_j} - \frac{1}{\rho_o} \frac{\partial}{\partial x_j} \left(\frac{\partial p}{\partial x_i} \right) - \frac{1}{3} \left(\frac{\partial(\frac{1}{\rho_o} f^i)}{\partial x_j} - \frac{1}{\rho_o} \nabla^2 p \right) \delta^{ij} \right] \quad (3)$$

In this equation ϖ_2 is a transport coefficient whose value is known for rigid spheres and Maxwell molecules as well, ρ_o and p_o are the equilibrium values for the density and pressure, respectively and p the local pressure $p(x^i, t)$. τ^{ij} in Eq. (1) is the total stress tensor, $\tau_{NS}^{ij} + \tau_B^{ij}$, where τ_{NS}^{ij} is given in Eq. (2). When we add these contributions, take their divergence and use the local equilibrium assumption, $p = p(\rho, T = const)$, so that $\frac{\partial p}{\partial x^i} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x^i} = C_s^2 \frac{\partial \rho}{\partial x^i}$ one finally gets that the linearized Burnett approximation to Eq.(1) is given by:

$$\begin{aligned} & \rho_o \frac{\partial u^i}{\partial t} + C_s^2 \frac{\partial(\delta\rho)}{\partial x_i} - 2\eta(\nabla^2 u^i - \frac{1}{3} \nabla_i(\nabla \cdot \vec{u})) \\ & + \frac{\varpi_2 \eta^2}{p_o} \left[\nabla^2 \left(\frac{f^i}{\rho_o} \right) - \frac{1}{3} \nabla_i \nabla \cdot \left(\frac{\vec{f}}{\rho_o} \right) - \frac{2}{3\rho_o} C_s^2 \nabla^2 \nabla_i(\delta\rho) \right] = f^i \end{aligned} \quad (4)$$

where C_s^2 is the speed of sound.

The linearization procedure, which is identical to the one followed in a previous publication [14], has clearly invoked that local state variables ρ and u^i deviate from its equilibrium values through linear fluctuations, namely

$$\rho = \rho_o + \delta\rho \quad (5)$$

and

$$u^i = u_o^i + \delta u^i = \delta u^i \quad (6)$$

The continuity equation now reads as

$$\frac{\partial(\delta\rho)}{\partial t} + \delta\theta = 0 \quad (7)$$

where $\delta\theta = \frac{\partial u^i}{\partial x^i}$. Eqs. (4)-(7) constitute the set of linearized coupled equations for $\delta\rho$ and u^i which can be solved for a given conservative force f^i . In our case of interest, the external force density is taken to be the gravitational force and, moreover, we assume that the fluctuating part of the gravitational field $\delta\varphi$ is governed by Poisson's equation

$$\nabla^2\delta\varphi = 4\pi G\delta\rho \quad (8)$$

where $G = 6.67 \times 10^{-11} \frac{m^3}{kg\cdot s^2}$ is the gravitational constant.

To eliminate u^i from this set we simply take the divergence of Eq. (4) and make use of Eq. (8) plus the fact that $\frac{\partial}{\partial t}\nabla \cdot = \nabla \cdot \frac{\partial}{\partial t}$ so that after a straightforward arrangement of terms we find that

$$\begin{aligned} & \frac{\partial^2(\delta\rho)}{\partial t^2} - C_s^2\nabla^2(\delta\rho) - \frac{8\pi}{3p_o}\varpi_2\eta^2G\nabla^2(\delta\rho) \\ & - \frac{4}{3}\frac{\eta}{\rho_o}\nabla^2\left(\frac{\partial(\delta\rho)}{\partial t}\right) + \frac{2}{3}\varpi_2\left(\frac{\eta^2}{p_o\rho_o}C_s^2\right)\nabla^2(\nabla^2\delta\rho) + 4\pi G\rho_o\delta\rho = 0 \end{aligned} \quad (9)$$

Eq. (9) is the main result in this calculation. It is a single equation describing the density fluctuations of an isothermal fluid when subject to the action of a fluctuating gravitational field taking into account all linearized contributions up to second order in the Knudsen parameter. Its implications on the Jeans number will be studied in the following section. Before we do so, however, it is interesting to point out the main features that in Eq. (9) originate from the Burnett approximation, those proportional to η^2 . The one of fourth order in the spatial derivatives, the next to the last term in the left hand side, corresponds to what was to be expected. But the other one namely, the third term in the left hand side shows a coupling between the gravitational field and the dynamics of the density fluctuations. This coupling is totally absent both at the Euler and the Navier-Stokes level. Its implications will be discussed later.

3 Analysis of the dispersion relation

In order to solve Eq. (9) we propose that the density fluctuations are represented by plane waves, so that

$$\delta\rho(\vec{r}, t) = Ae^{-i(wt - \vec{k}\cdot\vec{r})} \quad (10)$$

where the amplitude A needs not to be specified and the rest of the symbols have their standard meaning. Substitution of Eq. (10) into Eq. (9) yields:

$$\omega^2 + k^2(-C_s^2 + \frac{8\pi}{3p_o}\varpi_2\eta^2G\rho_o) + \frac{4}{3}\frac{\eta}{\rho_o}i\omega k^2 - \frac{2}{3}\varpi_2\frac{\eta^2C_s^2}{p_o\rho_o}k^4 + 4\pi G\rho_o = 0 \quad (11)$$

Eq.(11) is the sought dispersion relation from which we must examine the conditions leading to values of ω , necessarily complex, giving rise to runaway terms in Eq. (10). In fact, it may be rewritten as

$$\omega^2 + 2ia(k)\omega + f(k) = 0 \quad (12)$$

where

$$a(k) = \frac{2}{3}\frac{\eta}{\rho_o}k^2 \quad (13)$$

and

$$f(k) = (\frac{C_s^2}{\gamma} - \frac{8\pi}{3p_o}G\varpi_2\eta^2) - \frac{2}{3}\frac{C_s^2}{\gamma\rho_o}\frac{\varpi_2\eta^2}{p_o}k^4 + 4\pi G\rho_o \quad (14)$$

When examining the roots of this quadratic equation we see that the runaway solutions require that ω be complex and the threshold values for k to meet this requirement is that $a(k)^2 + f(k) \geq 0$. This condition reduces to a quadratic equation for the minimum value of k^2 namely,

$$r^2k^4 + sk^2 + 4\pi G\rho_o = 0 \quad (15)$$

where

$$r^2 = \frac{4}{9}\frac{\eta^2}{\rho_o^2} - \frac{2}{3}\varpi_2\eta^2G\rho_o \quad (16)$$

and

$$s = -C_s^2 - \frac{8\pi}{3p_o}\varpi_2\eta^2\frac{C_s^2}{\gamma\rho_o} \quad (17)$$

Calculating the roots of Eq. (15), taking the root with the positive sign, $k^2 > 0$, and expanding the square root we find, after a few trivial algebraic steps, that

$$\frac{k^2}{k_J^2} = 1 + \frac{\eta^2}{\rho_o}k_J^2\left(\frac{2\varpi_2}{p_o} - \frac{4}{9}\frac{\gamma}{C_s^2\rho_o}\right) \quad (18)$$

where the ordinary Jeans wave number is given by $k_J^2 = \frac{4\pi G\rho_o}{C_s^2}$. Eq.(18) is the sought result. In its derivation we have consistently neglected all terms of order η^4 and higher.

To what extent this result has an influence on the ordinary Jeans number will be discussed below. Two minor points should be noticed. Firstly, if $\varpi_2 = 0$ the corrective term, which is of order of k^4 is the one obtained within the Navier-Stokes approximation when ignoring the bulk viscosity [16]. Secondly, as we have pointed out initially, the Burnett linearized terms have to be included to have a consistent correction to order k^4 .

4 Discussion

As Eq. (18) clearly indicates, the inclusion of the Burnett linearized term provides a contribution to the correction of k_J^2 which is opposite in sign to the one arising from the Navier-Stokes equations. This may have a significant effect on the Jeans mass, which is inversely proportional to k_J^3 [6] [13] [15].

One may now ask about the conditions in which dissipative effects are indeed relevant in structure formation problems. In order to address this question we start with typical monatomic gases such as H, He, Ar, etc which we know, are reasonably described by a hard-sphere model whose shear viscosity extracted from kinetic theory is given by [4]:

$$\eta = \frac{5}{16\sigma^2} \sqrt{\frac{k_B m T}{\pi}} \quad (19)$$

Here σ is the atomic diameter, m the mass of the particles, k_B is Boltzmann's constant and T is the temperature of the system. The most relevant contribution will arise from the term within brackets. To estimate its value we recall that for hard spheres $\varpi_2 \simeq 2$ and the equation of state is $\rho_o = \frac{\rho_o R T}{M}$, where R is the universal gas constant and M the molecular weight of the gas. Then, the correction is determined by the expression

$$\hat{C} \equiv \frac{k^2 - k_J^2}{k_J^2} = \frac{4\eta^2 k_J^2}{\rho_o} \left(\frac{2\varpi_2}{\rho_o} - \frac{\gamma}{9C_s^2 \rho_o} \right) \quad (20)$$

Using the equation of state, the value of ϖ_2 and the fact that for a monatomic gas $C_s^2 = \frac{5}{3} \frac{k_B T}{M m_H}$, then for hydrogen $M = 2$ whence Eq.(20) reduces to

$$\hat{C} = \frac{8\eta^2 k_J^2}{R \rho_o T} \left(1 - \frac{\gamma}{15} \right) \quad (21)$$

Since $\gamma \sim 1.7$ for monatomic gases, this equation finally reads as

$$\hat{C} \simeq \frac{5}{8} \frac{GM}{\sigma^4} \frac{m_H^2}{R \rho_o T} \cong \frac{5}{\rho_o T} \times 10^{-19} \quad (22)$$

Eq. (22) has to be taken as a result which requires some care before conclusions may be extracted from it. As we said in the introduction, the Burnett regime is an adequate formalism to deal with fluids in the so called *transition regime*. This implies the regime prevailing between the rarefied regime and the continuous one. In the latter, the well-known Navier-Stokes equations of hydrodynamics provide an accurate description of the fluid. It appears that this transition regime could have prevailed in the Universe for temperatures $T \sim 10^4$ (K) and densities $10^{-15} \frac{kg}{m^3}$, but these figures are merely qualitative. If such conditions existed, then Eq. (22) simply asserts that structure formation would have been enhanced due to the coupling between hydrodynamic and gravitational modes. Notice that this coupling is the dominant term of Eq. (21). Clearly the importance of the correction critically depends on ρ_o and T . If $\rho_o T$

is larger than 10^{-19} as it probably occurred in earlier stages, the correction \hat{C} will be negligible so that, in order to reach a more definite statement, more reliable data is required. As a final comment, it should be mentioned here that in this calculation we have ignored taking into account the expansion in a co-moving reference frame. Whether or not this improves the results is a subject of further study.

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